Foundations of Quantum Theory in the Light of Bohmian Non-commutative Dynamics

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(2 April 2014)

The Finnish Society for Natural Philosophy 25 Years
K.V. Laurikainen Honorary Symposium 2013

Abstract

In this paper we discuss the notion of a 'particle' and its 'trajectory' in the Bohm approach to quantum theory in the context of a non-commutative dynamics which has classical mechanics as a commutative sub-algebra. We study the von Neumann-Moyal algebra and show that the key equations used in the Bohm approach are already contained within this structure. They appear as projections from the non-commutative phase space into space-time. This structure leads us to a more radical view of quantum processes involving individual particles.

1 Introduction

It gives me great pleasure to be part of a meeting to celebrate Professor Laurikainen's contribution to the philosophy of physics, particularly in the area of foundations of quantum mechanics. It was in this area that I first interacted with him a number of years ago. I was especially intrigued by his detailed study of the work of Pauli, which helped me understand better the contribution the Pauli made to the development of the early interpretations of quantum mechanics.

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Pauli not only contributed some profound papers on general relativity and quantum field theory, but influenced the debates on the interpretation of the quantum formalism, usually in the form of letters, which Laurikainen collected, enabling us to understand Pauli's unique contributions to those early debates.

My own particular interest at the time was in the appearance of quantum non-locality in entangled states. I was intrigued to understand why the absolute notion of locality was not questioned more explicitly when the EPR paradox was first proposed and discussed [14]. Pauli's letters seemed to suggest that he realised locality would have to be given up, but there was no real discussion of this issue that I have found. Perhaps Einstein's comments about "spooky action at a distance" was enough to stifle such a discussion. Nevertheless I was intrigued to find out if, after Pauli had started his collaborations with Carl Jung, quantum non-locality was behind his embracing Jung's notion of synchronicity. Although I have not found any discussions of giving up locality directly, nevertheless in the background of these discussions there was always the notion of "unus mundus", implying a kind of "wholeness" that Bohr considered to be one of the key new features of quantum phenomena.

Laurikainen was always willing to encourage debates on the meaning of the quantum formalism, but was unhappy with Bohm's 1952 proposals [2]. His main criticisms were two-fold. Firstly the formalism contained terms that were not accessible to measurement, namely the de Broglie-Bohm momentum and energy. Secondly the theory seemed to be a return to classical physics when Bohm adopts the classical canonical relations $p_B = \nabla S$ and $E_B = -\partial_t S$ but replaces the classical action, S, by the phase of the wave function $\psi = Re^{iS/\hbar}$ without justification. Nevertheless by making this change it has been shown that the notion of a particle trajectory can be consistently maintained if the formalism is used correctly. This directly contradicted Bohr's assertion that it was no longer possible to give classcial-like pictures of quantum phenomena because this implies that a system under investigation could be sharply distinguished from the means of observation. For Bohr the impossibility of making such a sharp separation was the key notion that implied the new radical notion of wholeness.

Secondly the criticism raised in some philosophical circles claims that Bohm could only maintain the classical notion of a "particle trajectory" because he had changed the mathematical content of the theory by *adding* a new term. This false conclusion was probably encouraged by Heisenberg's criticism [25] of Bohm's proposal when he writes ".... some strange quantum potentials introduced *ad hoc* by Bohm". The extra term that Heisenberg was referring to is not "added" *ad hoc*. It appears when one derives the real part

of the Schrödinger equation under polar decomposition of the wave function and has not been added.

These two features reveal a profound misunderstanding of the whole approach. Recently it has been realised that the terms like P_B and E_B are weak values and these can now be measured using new experimental techniques [28, 31, 32]. Although concepts of "trajectories" can still be maintained, they arise from the quantum formalism itself, without the need to add any new terms and without the need to appeal in any way to classical mechanics. Classical mechanics is to be seen as a limiting case of the quantum motion. The key question concerns the nature of this deeper structure, which I will discuss in detail in this paper.

2 Return to the Origins of Quantum Theory

2.1 Non-commutative Products

Rather than developing my ideas from some contemporary interpretation, I would like to go back into the history of quantum mechanics and to re-examine the original, somewhat neglected Weyl-von Neumann approach of the 1930s. This approach is radically different from the approach that appears in von Neumann's classic book "Mathematical Foundations of Quantum mechanics" [38]. Just before the book appeared, von Neumann wrote a classic paper "Die Eindeutigkeit der Schrödingerschen Operatoren" [37], which suggests a very different approach to quantum phenomena than the one set out in his book. This paper showed, among other things, how the whole operator formalism could be replaced by functions on a non-commutative symplectic space bringing us into the realm of a modern approach to geometry. Formally we write

$$\widehat{A} \Leftrightarrow A(\alpha, \beta) \tag{1}$$

Here $\widehat{A}$ is an operator in a Hilbert space and $A(\alpha, \beta)$ is a function on some symplectic space. The equivalence between the two approaches follows from the relation

$$\widehat{A} = \int\int A(\alpha, \beta)\widehat{S}(\alpha, \beta) d\alpha d\beta \tag{2}$$

where $\widehat{S}(\alpha, \beta) = e^{i(\alpha\hat{P} + \beta\hat{X})}$. For simplicity, we will only consider a two dimensional symplectic space as the generalisation to $2n$-dimensions is straight forward and adds no new features that are relevant to our discussion, which is necessarily an over-view, emphasising the main points.

The significance of the symplectic space is that it is the space within which classical mechanics can also be formulated, so that both quantum and classical mechanics appear in the same space, the difference being that in classical mechanics the product of the functions is commutative, while in the quantum domain, the functions must satisfy a non-commutative product, $F(\alpha,\beta)\star G(\alpha,\beta) \neq G(\alpha,\beta)\star F(\alpha,\beta)$. This product contains the classical structure as a sub-structure as we will show below. This generalised product is now known in the physics community as a Moyal product even though it was first defined in the von Neumann paper eighteen years earlier! To the mathematicians the product is known as a twisted product [22]

Moyal's contribution was twofold. Firstly, he suggested that we write $\alpha = x$ and $\beta = p$, thus attempting to give a physical significance to the abstract formalism of von Neumann, namely that it is some form of "physical phase space", albeit a non-commutative version. Secondly, he suggested that the resulting formalism could be regarded as a generalisation of ordinary statistical theory where we replace a key feature of this theory, the characteristic function, by a *characteristic operator*. In this way a classical statistical phase space becomes a special case of a non-commutative statistical theory. In fact the non-commutative phase space contains a commutative subspace that *is* the classical phase space. This commutative limit appears when we neglect terms of $O(\hbar^2)$. This is exactly the type of structure that Primas and Müller-Herold [35] have called for.

Let us immediately dismiss one worry that always arises at this point. Surely we cannot have an (x,p) phase space in the quantum domain because of the uncertainty principle. The problem here is that in changing from α to x and β to p, one immediately assumes the symbols (x,p) mean the position and momentum of a *point* particle. But a deeper investigation shows that the x and p are coordinates of the centre of an extended region of phase space, a region that de Gosson has called a 'blob' [21]. A similar extended notion of a particle was also proposed by Weyl [44] when he writes'

> Hence a particle itself is not even a point in field space, it is nothing spatial (extended) at all. However, it is confined to a spatial neighbourhood, from which its field effects originate.

This idea of an extended region in phase space becomes much more compelling if we evoke the "no squeezing theorem" of Gromov [23]. This deep theorem reveals a new topological invariant that arises even in a commutative symplectic geometry. It can be regarded as the "footprint" of the uncertainty principle that quantum mechanics leaves in the classical domain [19, 20]. It is only in the classical limit that the 'blob' becomes a point particle. Thus to make the whole formalism work, we are forced to regard the 'particle',

not as a point, but as a 'blob' with a finite extension in phase-space. Indeed to accommodate this structure we need a product that is translation and symplectic covariant, associative and *non-local* as was emphasised by Várilly and Gracia-Bondia [42]. As we will see it is this non-local feature that mathematically captures Bohr's notion of wholeness.

2.2 Non-commutative Probability.

Of the two suggestions of Moyal discussed in the previous sub-section, the most important one for our purposes here was his proposal, that in quantum physics, we are dealing with a generalisation of an ordinary statistical theory to a non-commutative one. What Moyal noticed was that the element $\widehat{S}(\widehat{P},\widehat{X}) = e^{i(\alpha\hat{P}+\beta\hat{X})}$, used in equation (2), behaves like an *operator* generalisation of the 'characteristic function'. It is this function that plays a central role in ordinary statistical theories. The corresponding characteristic function is defined as an expectation value of this operator $\widehat{S}(\widehat{P},\widehat{X})$, giving

$$F_\psi(\alpha,\beta,t) = \int \psi^*(x,t)e^{i(\alpha\hat{P}+\beta\hat{X})}\psi(x,t)dx.$$

Then the expectation of any operator $\widehat{A}$ can be written in the form

$$\langle\widehat{A}\rangle = \int\int a(\alpha,\beta,t)F_\psi(\alpha,\beta,t)d\alpha d\beta. \tag{3}$$

This suggests that $F_\psi(\alpha,\beta,t)$ can be treated as probability measure in the non-commutative symplectic geometry. Historically the feature was rejected because the probability measure can take on negative values inspite of the implicit support of both Dirac [11] and Bartlett [1] in the '40s. They argued that there was nothing wrong with using negative measures and offered an interpretation of these measures. However one should notice that although the probability measure can take negative values, the probability values of observable quantities always turn out to be positive as was shown by Feynman [16].

2.3 Non-commutative Symplectic Geometry.

The precise form of non-commutative product in the symplectic space follows from

$$F(\alpha,\beta) = G(\alpha,\beta)\star K(\alpha,\beta) = \int\int e^{2i(\gamma\beta-\delta\alpha)}G(\gamma-\alpha,\delta-\beta)K(\alpha,\beta)d\alpha d\beta,$$

a relation first derived by von Neumann. It is not difficult, but tedious to show this product is non-commutative (for details see von Neumann [37]).

Moyal obtained a simpler expression for this product, which he wrote in the form

$$G(p,x,t) \star K(p,x,t) = G(p,x,t) \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overrightarrow{\partial}}{\partial x}\frac{\overleftarrow{\partial}}{\partial p}\right)\right] K(p,x,t).$$

Here we have already written $\alpha = x$ and $\beta = p$. If we choose $G(p,x,t) = x$ and $K(p,x,t) = p$, this expression immediately gives

$$x \star p - p \star x = i\hbar$$

illustrating how the Heisenberg commutator is reflected in the star product of phase space.

The Moyal form of the star product immediately enables us to see how the classical commutative phase space emerges from this structure. First we define two brackets, the Moyal bracket and the Baker bracket (or Jordan product). These are defined by

$$\{G,K\}_{MB} := \frac{G \star K - K \star G}{i\hbar} \quad \text{and} \quad \{G,K\}_{BB} := \frac{G \star K + K \star G}{2}$$

The important feature of these brackets is that they can be expanded as polynomials in $\hbar$. We then find to order $\hbar^2$, the Moyal bracket reduces to the Poisson bracket, while the Baker bracket reduces to the commutative product $G(x,p)K(x,p)$. Thus the von Neumann-Moyal algebra contains the classical limit. It is this relationship from which deformation quantisation arises [30].

There is no need to look for a correspondence between the quantum commutator and the Poisson bracket, a process that fails as was demonstrated in the Groenwald-van Hove theorem [24]. Furthermore there is no need to introduce the notion of decoherence as a fundamental process to obtain classical physics. Of course, this does not mean that decoherence plays no role in quantum processes. It plays a vital role in real experiments where noise and other thermal processes enter the experiment to destroy quantum coherence.

3 Time Development Equations in a Non-commutative Structure.

3.1 Background.

Given that the theory we are developing is a non-commutative statistical theory, we are now faced with the problem of finding the time development

equations in a non-commutative algebra. This means that we must have two different types of elements, those acting from the left and those acting from the right, so that left and right translations must be treated separately.

Such a structure was already anticipated by Dirac [10] in a different context. He noticed that in trying to bridge the gap between those problems involving only pure states, $|\psi\rangle$, and those involving mixed states, which require a density matrix, $|\psi\rangle\langle\phi|$, we are faced with a similar problem, namely the possibility of two-sided differentation.

To see what this means, suppose we replace $|\psi\rangle$ by a matrix of n rows and m columns, n being the number of rows and columns in the square energy matrix H, which we can think of as representing the Hamiltonian. Clearly this must satisfy an equation that has the same form as the Schrödinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

Suppose $\langle\phi|$ is replaced by a matrix with m rows and n columns. Here H must act on this from the right,

$$-i\hbar\frac{d\langle\phi|}{dt} = \langle\phi|H.$$

Now the product $|\psi\rangle\langle\phi|$ will be replaced by a square matrix with n rows and n columns and will satisfy

$$\begin{aligned} i\hbar\frac{d(|\psi\rangle\langle\phi|)}{dt} &= i\hbar\frac{d|\psi\rangle}{dt}\langle\phi| + |\psi\rangle\left(i\hbar\frac{d\langle\phi|}{dt}\right) \\ &= H|\psi\rangle\langle\phi| - |\psi\rangle\langle\phi|H. \end{aligned}$$

which, of course when written in the form

$$i\hbar\frac{d\rho}{dt} = H\rho - \rho H = [H,\rho] \tag{4}$$

is just the quantum Liouville equation. To arrive at this equation we have subtracted the Schrödinger equation from its dual.

The question that has interested me is to ask what happens if we *add* the Schrödinger equation to its dual. We will immediately get

$$i\hbar\left(\frac{d|\psi\rangle}{dt}\langle\phi| - |\psi\rangle\frac{d\langle\phi|}{dt}\right) = H\rho + \rho H. \tag{5}$$

At first sight this equation looks unfamiliar, but it may be helpful to write the LHS in terms of a standard notation used in field theory [36], namely,

$$\left(\frac{d|\psi\rangle}{dt}\langle\phi| - |\psi\rangle\frac{d\langle\phi|}{dt}\right) = |\psi\rangle\overleftrightarrow{\partial}_t\langle\phi|$$

This is recognised as a symbol used in relation to the energy of the Dirac electron. Here the left and right translations have a profound influence on the structure and this form was used by Schwinger [41] in his classic paper that laid the foundations of QED.

This extra equation is not well known but it has appeared explicitly in the chemical literature, as for example in Dahl [8]. If we are relying on the quantum Liouville equation to describe our system, as we do when we are dealing with mixed states, then we need the extra equation (5) to complete the specification of the energy properties of the mixed state.

Although Dirac's aim was to find the relationship between the way we deal with mixed states using matrices and the way we deal with pure state vectors, one should realise that we can also apply equations (4) and (5) to a system in a pure state. For such a state, $\rho^2 = \rho$ and if the matrix is of rank one, then and only then can it be written as $|\psi\rangle\langle\phi|$, where these symbols now take on their usual meaning. Of course it may be argued that it seems perverse to propose more complicated method when the straight forward method of using the Schrödinger equation is standard. However I am interested in the relationship between the different approaches to the *same mathematical structure* and to explore this relation in detail I need this more general formulation.

At this stage it is useful to bring out the relationship between this seemingly abstract structure and the equations used in the Bohm interpretation for if I do not mention Bohm then I will be accused of losing my way!

Equations (4) and (5) are representation free, so let us examine equation (5) in the x-representation. This leads directly to the quantum Hamilton-Jacobi equation that forms the basis of the Bohm approach.

$$i\hbar\frac{dS}{dt} + \frac{1}{2m}\left(\frac{dS}{dx}\right)^2 + Q(x) + V(x) = 0 \tag{6}$$

where $V(x)$ is the classical potential and $Q(x)$ is an additional term which has been called the quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{\partial^2 R}{\partial x^2}.$$

where we have again written $\psi = R\exp[iS/\hbar]$. If we regard $Q(x)$ as a qualitatively new form of energy appearing at the quantum level, then this equation, and in consequence equation (6), are expressions of the conservation of energy.

This method of arriving at equation (6) should be contrasted with the original approach of Bohm [2] who simply arrives at the equation by looking

at the real part of the Schrödinger equation when the wave function is written in polar form. Here however, equation (6) appears as a special case of a more general approach. Indeed this approach can then also be applied to the Pauli and Dirac equations [26].

3.2 von Neumann-Moyal Algebra

Let us now derive the time development equations in the full von Neumann-Moyal algebraic approach introduced in section 3. If we again write $\alpha = x$ and $\beta = p$, we find

$$H(x,p) \star F_\psi(x,p,t) = i\hbar \int e^{-i\tau p} \psi^*(x - \hbar\tau/2) \overrightarrow{\partial}_t \psi(x + \hbar\tau/2) d\tau. \tag{7}$$

and

$$F_\psi(x,p,t) \star H(x,p) = -i\hbar \int e^{-i\tau p} \psi^*(x - \hbar\tau/2) \overleftarrow{\partial}_t \psi(x + \hbar\tau/2) d\tau. \tag{8}$$

Subtracting equation (8) from equation (7) gives us a Moyal bracket equation

$$\partial_t F_\psi = (H \star F_\psi - F_\psi \star H)/i\hbar = \{H, F_\psi\}_{MB}. \tag{9}$$

If we expand this equation to $O(\hbar^2)$, we arrive at the classical Liouville equation

$$\partial_t F + \{F, H\}_{PB} = 0, \tag{10}$$

where $\{..\}_{PB}$ is the Poission bracket. As we have remarked above, the classical limit appears as a sub-agebra and the Moyal bracket equation gives an expression for the conservation of probability.

There is also a second equation, the Baker bracket equation, formed by adding equations (7) and (8). This is

$$2\{H, F_\psi\}_{BB} = i\hbar \int e^{-i\tau p} [\psi^*(x - \hbar\tau/2) \overleftrightarrow{\partial}_t \psi(x + \hbar\tau/2)]. \tag{11}$$

At first sight it is not clear what the RHS of equation (11) means, but it turns out that it is an expression for the conservation of energy. We can get a clearer insight into this by again writing $\psi = Re^{iS/\hbar}$. We find

$$\frac{\psi^* \overleftrightarrow{\partial}_t \psi}{\psi^* \psi} = \left[\frac{\partial_t R(x + \hbar\tau/2)}{R(x + \hbar\tau/2)} - \frac{\partial_t R(x - \hbar\tau/2)}{R(x - \hbar\tau/2)} \right] + i \left[\frac{\partial_t S(x + \hbar\tau/2)}{S(x + \hbar\tau/2)} - \frac{\partial_t S(x - \hbar\tau/2)}{S(x - \hbar\tau/2)} \right]$$

which still does not look too helpful. However if we go to the limit $O(\hbar^2)$ we find

$$H \star F_\psi + F_\psi \star H = -2(\partial_t S)F_\psi + O(\hbar^2)$$

By substituting this expression into equation (11) we find

$$(\partial_t S)F_\psi + HF_\psi = 0.$$

By cancelling out the F_ψ, we are left with the classical Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H = 0. \tag{12}$$

Once again we see a different aspect of the classical limit emerging.

Note that equations (9) and (11) are equations arising from within the von Neumann quantum algebra and therefore is fully quantum mechanical. In fact these equations contain *more* information than equations (4) and (9) which are written in terms of the usual bra-ket notation. To see why this is so, we begin by simply remarking that equations (9) and (11) are equations operating in a phase space (i.e., a symplectic space), whereas equations (4) and (5) are more suitable for a description in configuration space. Clearly the space involved in the von Neumann-Moyal approach has twice the number of dimensions so it is already a larger space.

Since the symplectic phase space has double the number of dimensions of the configuration space there must be some way of introducing a projection operator to relate the two spaces. The formal introduction of this projection operator is too technical to present here. However a full discussion of the existence of this projection operator can be found in Varilly and Gracia-Bondia [42]. A simpler account is outlined in Hiley [29] .

I should like to end this section by saying a few more words about the relation between the algebraic structure and the traditional approach to quantum mechanics. In his original paper, Schrödinger [40] started from the classical Hamilton-Jacobi equation and made an attempt, as he put it, to find a Hamiltonian "undulatory mechanics" based on a generalisation of the symplectic structure of Hamiltonian ray optics. His pioneering attempt was heuristic, and at a certain step in his 'derivation', he added a footnote stating "I realise that this formulation is not quite unambiguous". So although he arrived at an equation which gave correct answers, it is not clear where the equation actually come from[1]. The equation is generally presented in a heuristic way

[1]Feynman's answer to the question "Where does the equation come from?" was, "Nowhere. It's not possible to derive it from anything we know. It came out of the mind of Schrödinger..." [15].

and justified *a posterori*. However in a recent paper, de Gosson and I have shown that the mechanics he was looking for can be found in the covering group of the symplectic group, the metaplectic group [20]. It is the covering group that the essential features of quantum phenomena find their home, but the importance of the covering group is another story

3.3 Relation to the Density Matrix

Let us return to consider the expectation value of the characteristic operator, $F_\psi(\alpha, \beta, t)$, in more detail. As we have pointed out, although Moyal identified α with x and β with p, he gave no reason to identify these parameters with the position and momentum of a localised particle.

Bohm and Hiley [4] developed an algebraic approach to a generalised phase space from a different point of view. Following the suggestion by Frölich [17] that the density matrix should be regarded as a fundamental description of state of a system, they started from a two-point density matrix, $\rho(x_1, x_2, t)$ in *configuration* space and, using the Wigner transformation, obtained an expression for $F(x, p, t)$ which had the same form as used in equation (3), but where the (x, p) now correspond to the mean values $x = (x_2 + x_1)/2$ and $p = (p_2 + p_1)/2$ of two points in phase space. The detailed calculations show that it is this non-local feature that is ultimately responsible for the non-commutative product. The pair of points are conjugate points and in consequence the parameters (x, p) defined in this way are not subject to the uncertainty principle because the operator equivalents commute. That this feature gives rise to the non-locality in the Moyal product has been formally proved by Varilly and Gracia-Bondia [42].

Finally let us show that the Moyal algebra contains the two defining equations of the Bohm approach. Since we are assuming $F_\psi(x, p, t)$ is a probability distribution, we can construct two conditional expectation values. The first is the Moyal momentum, P_M, defined by

$$\rho(x)P_M = \int pF_\psi(x, p)dp = \frac{\hbar}{2i}[\psi^*\nabla\psi - (\nabla\psi^*)\psi] = \nabla S.$$

This is identical to the Bohm momentum $P_B = \nabla S$. Traditionally this has been regarded as a guidance condition, but in this approach there are no guiding waves, simply dynamics. Furthermore Moyal derives an equation for the transport of this momentum, which turns out to be exactly the quantum Hamilton-Jacobi equation (6). Thus we see there exists a very close relationship between the Moyal algebra and the approach of Bohm. In this view, the Bohm momentum P_B is simply the conditional expectation value of the momentum p. Thus the Bohm approach itself can be considered as a

consequence of the generalised statistics. But even more, it is an aspect of a generalised non-commutative dynamics.

4 Interpretations

We must now turn to examine the question of what this all means as far as an interpretation of the formalism is concerned, a question that would have interested Professor Laurikainen. In the algebraic approach the 'wave' no longer plays a dominant role, nor, incidentally, does the point particle. Process becomes the basic descriptive form with the wave function being replaced by a density element, $F(x, p, t)$. This function characterises an extended region of phase space containing the energy involved in the process, de Gosson's 'quantum blob' [9, 21]. Remember this is for a single excitation; with entangled particles, the energy becomes even more non-local.

The time evolution of this region is governed by a non-commutative dynamics described by equations (7) and (8). In the classical limit these equations become the classical Louiville equation and the classical Hamilton-Jacobi equation which, as is well known, are key equations of classical dynamics. In this limit the region of phase space behaves like a 'point' so the classical particle reappears. However in reality all particles have an internal structure and the stability of the particle is maintained by an internal quantum dynamics.

As we have seen, with non-commutativity comes non-locality so that, in the quantum domain, we no longer have point-like particles but regions of quasi-local energy in the case of single particles. How then are we to interpret Bohmian trajectories? The parameter x gives the mean position of the quantum blob, and we can take $P_B(x)$ to describe its mean momentum with which it is transported by the quantum Hamilton-Jacobi equation (6). This enables us to retain the notion of a quasi-localised particle, following a trajectory.

Note that what we are calling a 'particle' is an invariant feature arising from the structure of the two real fields and is not added as an afterthought. In other words we must not identify this quasi-localised particle as a separate 'rock-like classical particle'. However it is still a beable in the sense that it identifies a single individual quantum process, but this process in not point-like and must not be identified with a classical particle. In this way we can argue that we have an individual process, the mean value of which, moves along a trajectory. If we do this then we are not lead to any contradictions or inconsistencies in interpreting the results of experiments, but in doing this we must not introduce other classically plausible features that are not consistent

with the mathematical formalism.

I could end the story here but there is another possibility within the algebra we can consider. $F(x, p, t)$ is symmetric in x and p so that we can also consider the conditional expectation value for the *position*, X_M. Now the process can be characterised by a different pair of variables (X_B, p) so that we are able to construct another phase space based on these particular variables. Again we have the possibility of defining flow lines or trajectories in this space, but these can be very different from the previous flow lines as shown in Brown and Hiley [7]. We seem to be back at the position summarised in the following quotation of Pauli [39]

> One can look at the world with a p-eye or with an x-eye, but if one wishes to open both eyes at the same time, one goes wrong.

Thus we are faced with a dilemma, which of these two phase spaces is the 'real' phase space? If we assume that space-time is an absolute necessity, say, if only because all our experiments are performed in space-time, then we can follow Dürr and Teufel [12] and argue that all physical processes, quantum and classical, must evolve in space-time: this is our reality. Therefore we must take the phase space constructed on (x, P_B) to be the correct description of all physical processes.

But why should space-time be primary? It seems the simplest thing to do, but what is the compelling reason for such an assumption? Kant certainly assumed that space was an *a priori* given, but was he right?

Let us try to look at it in a different way. All our knowledge of the properties of space-time are abstracted from the behaviour of physical processes. For example, with the rigid metre rule, we construct Euclidean geometry. With a radar gun and a clock, Page [33] constructs the Minkowski space-time. In general relativity we find a curved space-time conditioned by the presence of matter itself. Space-time is not a given *a priori*, it is abstracted or even constructed from physical processes.

We can see this close tie between physical processes and geometry in a mathematical construction due to Gelfand-Naimark [43]. What this construction shows is that if we are faced with a commutative dynamical algebra, there is a duality between the algebra and the manifold that supports that dynamics. We can either follow the traditional route by starting from an *a priori* given space-time with its topological and metrical structure given and then build on it the dynamics of physical processes. Alternatively we can start from the commutative dynamics defined by the physical processes themselves and then abstract from this structure a unique geometry; its points appearing as maximal ideals, together with well defined topological and metrical properties. Einstein's commutative general relativity shows that space

and its properties are co-defined. A discussion of how general relativity fits in to the Gelfand-Naimark scheme can be found in Geroch [18].

When we are faced with a non-commutative dynamics, we have a serious problem. There is no unique underlying manifold, no unique underlying phase space and therefore no unique underlying space-time. Such a situation was already anticipated by Einstein [14] when he wrote

> ....perhaps the success of the Heisenberg method points to a purely algebraic description of nature, that is, to the elimination of the continuous functions from physics. Then, however, we must give up, in principle, the space-time continuum.

Einstein here attributes the non-commutativity to Heisenberg. The von Neumann-Moyal algebra builds Heisenberg's non-commutativity into a non-commutative symplectic phase space.

When we consider a non-commutative geometry, we do not have a unique underlying manifold, but we can construct *shadow manifolds*. The phase space manifolds (x, P_B) and (X_B, p) are just examples of these shadow manifolds. This is essentially the same situation anticipated by Pauli [34] quoted above.

The reason why one thinks this is not going in the right direction is, perhaps, we expect the quantum process to be described in terms of a single unique classical phase space. In other words we are expecting to map everything into a single space. That would be fine if we were standing outside the phenomena, but we are not outsiders, we are part of the whole process. We may be detached observers but our measuring instruments are 'inside' the process; they are part of the total process. This fits the essential message that is contained in Bohr's notion of wholeness as he insisted

> The crucial point,..., implies the impossibility of any sharp separation between the behaviour of atomic objects and the interaction with measuring instruments which serve to define the conditions under which the phenomena appear.

He continues

> Consequently, evidence obtained under different experimental conditions cannot be comprehended within a single picture but must be regarded as *complimentary* in the sense that only the totality of the phenomena exhausts the possible information about the objects. [Bohr [6]]

The non-commutative algebra gives a mathematical expression to Bohr's words. The two shadow phase spaces are not in competition with each other, but are necessary to give a more complete description of the quantum process, which is what Bohm called 'implicit'. Thus the algebraic approach provides a deeper understanding of quantum phenomena and is *not* to be treated as a rival alternative.

5 Conclusion

We have shown in this paper how the non-commutative quantum algebra introduced by von Neumann has contained with in it both the Moyal and the Bohm interpretations. They are merely different aspects of the same non-commutative algebra. Furthermore this algebra has within it a commutative sub-algebra which contains classical mechanics. A theory that discusses the possibility of smooth transitions between the quantum and the classical theories is called deformation quantum mechanics [30].

We have also shown how Bohr's principle of complementarity and his notion of wholeness, the inseparability of the object from its means of observation, are contained within the algebra. The time development equations describing the quantum process do not need 'wave' functions for their definition. The 'wave' is just a heuristic device to motivate an interest in the formalism without the need to go into the deeper mathematical background. It should not be given importance by calling it a state function. That leads to the apparently insolvable problems of the interpretation of the formalism in this form.

In the algebraic approach, it is the density matrix $F(x, p, t)$ that plays the central role. This object is an element in the algebra itself and does not have to be imported from outside of the algebra. This is a more general approach in which the ordinary approach emerges when the density matrix is idempotent and of rank one. It is then that the wave function emerges. Since the wave function is now given a secondary role, there is no measurement problem and measurement is handled in the same way as it is in the Bohm approach [5].

On the philosophical side, this non-commutative algebra is actually a mathematical expression of Bohm's implicate order [3]. The algebra is a mathematical description of what Bohm calls the implicate order. The shadow manifolds are examples of what Bohm calls explicate orders arising from the participation of ourselves or our measuring instruments in the process itself. The implicate order takes process as it basic starting point and from a set of structuring principles, one constructs, not only particles,

but space-time itself. We are unable to develop these ideas further here, but an account of this approach and its relation to our discussion here can be found in Hiley [27].

Acknowledgments

I should like to thank Viljo Martikainen and the Finnish Society for Natural Philosophy for their invitation and support to attend their conference 25 years K.V. Laurikainen Honorary Symposium. I should also like to thank Maurice de Gosson, Paavo Pylkkänen and members of the TPRU for their help in clarifying ideas contained in this paper.

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